On the median residual lifetime and its aging properties

Rosa E. Lillo
Dpto. de Estadística
Universidad Carlos III de Madrid
Madrid. Spain
lillo@est-econ.uc3m.es

Abstract

The concept of median residual lifetime for a probability distribution is analyzed from the point of view of reliability. Two criteria for aging are studied using the median residual lifetime instead of the mean residual lifetime. Relations with other seven classical criteria for aging are provided. Finally, it is proved that there is not a relation between the behavior (increasing or decreasing) of the median residual life time and the mean residual life time.

1 Introduction

In life–testing situations, given that a component has survived until time t, the mean additional lifetime is called the mean residual life function (MRLF). More specifically, if X is the life of a component, then

$$\mu(t) = E\left(X - t \,|\, X > t\right), \qquad \text{for all } t \geq 0$$

is the MRLF. The MRLF has been extensively employed in the reliability literature (see Watson and Well (1961), Bryson and Siddiqui (1969) and Muth (1977)) and in the social sciences, the common empirical phenomenon of an increasing MRLF has been referred to as "inertia" and its presence in the data on duration of jobs, strikes and wars has been considered in the literature. It is well known (see Gupta (1975), Hall and Wellner (1981) and Lillo and Martín (1999)) that the MRLF determines the distribution function uniquely.

Schmittlein and Morrison (1981) point out that the MRLF has a number of practical shortcomings, especially in situations where the data are censored. In such cases the empirical mean residual life cannot be calculated. Moreover, even in the case of complete data, the estimated mean residual life will tend to be unstable due to its strong dependence on the very long durations. As an alternative, they recommend the median residual life function $\alpha(t)$ representing the median additional time to failure given no failure by time t. This conditional distribution is a distributional feature of considerable interest in modeling survival and reliability data. Calculation of this quantity poses no difficulty as long as one is able to record half of the observations. Besides, there is an increasing number of papers using median regression models instead of mean regression models in which the concept of median residual life function appears (see Ying et all (1995), Kottas and Gelfand (2001)). This paper is focus on studying different aspects of the median residual life function such as applications in Reliability, relations with the MRL and properties shared with the associated survival function.

If $R(t) = P(X > t)$ is the reliability function associated to the random variable X, the median residual life function α is

$$\alpha(t) = R^{-1}\left(\frac{1}{2}R(t)\right) - t, \qquad \text{for all } t \geq 0.$$

Then, $\alpha(t)$ verifies the equation,

$$R\left(t + \alpha(t)\right) = \frac{1}{2}R(t), \qquad \text{for all } t \geq 0. \tag{1}$$

Gupta and Langford (1984) proved that, given $\alpha(t)$, the solution (1) is not unique and the median residual life function does not characterize the distribution function as the mean residual life function. On the other hand, Zoroa (1973) also showed that the median residual life function does not characterize the distribution function, but a complete solution to (1) was not given in that reference.

In this work, a characterization property of the median residual life function is proved, which will be a useful tool to establish relations between analytic properties, such as differentiability or convexity of the survival function, and the associated median residual life function. Bryson and Siddiqui (1969) showed how seven criteria for aging are related, being the MRLF present in two of them. We prove that the same pattern is followed if the MRLF is substituted by the median residual life function. Finally, it is showed that there is no relation between the aging criteria: increasing/decreasing MRLF and increasing/decreasing median residual life function. This implies the importance of the use of the median residual life function in modeling survival and reliability data.

2 Characterization property of the median residual life function

Let $a(t) = \alpha(t) + t$ denoted as the median life function. Given a positive random variable X which has a strictly increasing, continuous (but not necessarily absolutely continuous) cumulative distribution function, then its associated median life function $a(t)$ is continuous, strictly increasing, maps $[0, \infty)$ into itself, and satisfies $a(t) > t$ for every $t \geq 0$. The result of Gupta and Langford (1984) shows that the median residual life function does not characterize the distribution function but a family of distributions functions. In the following result, the objective is to give a different approach for that result, based on the construction of the distribution function by means of both its median residual life function and the survival function on the interval $(0, \alpha(0))$.

Theorem 1 *Let $a(t)$ be a continuous, strictly increasing function that maps $[0, \infty)$ into itself, and satisfies that $a(t) > t$ for every $t \geq 0$. Let G be a strictly decreasing and continuous function that maps $[0, a(0)]$ into $[1/2, 1]$ such that $G(0) = 1$. Then, an unique survival function R exists such that its related median life function is $a(t)$ and $R(t) = G(t)$ for all $0 \leq t \leq a(0)$.*

In order to characterize analytic properties of $a(t)$, it is of interest to investigate the relation between properties such as differentiability or convexity of the survival function with the same properties in $a(t)$, and conversely. In this work, we provide some results showing implications of this type. Besides, some questions suggested in the paper of Gupta and Langford (1984) related to the characterization of the Pareto distribution through its linear median residual function, plus some analytic properties of the survival functions as convexity, are answered.

3 Criteria for system aging using the median residual life function

The concept of *aging* or progressive shortening of an entity's residual life time, is discussed in terms of the entity's survival time distribution. Quantities defined to describe the aging phenomenon include the *specific aging factor*, *hazard rate*, *hazard rate average*, and *mean residual life time*. We refer to Barlow and Proschan (1975) for the usefulness of these concepts in reliability theory. Bryson and Siddiqui (1969) established a set of seven criteria for aging based on these quantities, and a chain of implications among the criteria. The aim in this Section is to provide the same chain of implications as in Bryson and Siddiqui (1969) but using the median residual life time instead of the MRLF. First, we introduce the definition of the quantities implied in the criteria for aging.

Let X be a random variable denoting system lifetime. The survival time distribution is denoted by R, being $R(t) = 1 - F(t)$, where $F(t)$ is the c.d.f. We shall assume that the system is functioning at time $t = 0$ and that it will fail or die at some $t > 0$, so that $R(0) = 1$ and $R(\infty) = 0$. Also, we assume differentiability of $R(t)$, with $f(t) = -R'(t)$, denoting probability density in the usual manner. With this notation, we have the following definitions that are motivated in Bryson and Siddiqui (1969).

- The *hazard rate* corresponding to a survival function $R(t)$ is

$$h(t) = \frac{f(t)}{R(t)}, \qquad \text{for all } t \geq 0.$$

- The *specific aging factor* of a system at time t, specific with respect to a positive time parameter s, is defined as

$$A(t,s) = \frac{R(t)R(s)}{R(t+s)}, \qquad \text{for all } t,s \geq 0.$$

- The *specific interval–average hazard rate* is

$$H(t,s) = \frac{\int_s^{s+t} h(x)dx}{t}, \qquad \text{for all } t,s \geq 0,$$

  where $h(x)$ is the hazard rate corresponding to $R(t)$.

With these definitions, Bryson and Siddiqui (1969) constructed several criteria for aging of a system.

Criterion 1. Increasing specific aging factor.

$$A(t_2,s) \geq A(t_1,s), \qquad \text{for all } s \geq 0,\ t_2 \geq t_1 \geq 0.$$

Criterion 2. Increasing hazard rate (IHR).

$$h(t_2) \geq h(t_1), \qquad \text{for all } t_2 \geq t_1 \geq 0.$$

Criterion 3. Increasing interval–average hazard rate.

$$H(t_2,s) \geq H(t_1,s), \qquad \text{for all } s \geq 0,\ t_2 \geq t_1 \geq 0.$$

Criterion 4. Decreasing mean residual lifetime.

$$\mu(t_2) \leq \mu(t_1), \qquad \text{for all } t_2 \geq t_1 \geq 0.$$

Criterion 5. Increasing hazard rate average (IHRA)

$$H(t_2,0) \geq H(t_1,0), \qquad \text{for all } t_2 \geq t_1 \geq 0.$$

Criterion 6. Positive aging.

$$A(t,s) \geq A(0,s), \qquad \text{for all } t,s \geq 0.$$

Criterion 7. Net decreasing mean residual lifetime.

$$\mu(t) \leq \mu(0), \qquad \text{for all } t \geq 0.$$

Bryson and Siddiqui (1969) proved that the seven criteria are related by means of the pattern given in Figure 1, in which each arrow indicates implication in the direction shown. In this work, we show that the same pattern holds when the mean residual life function is replaced by the median residual life function, that is, Criteria 4 and 7 are substituted by Criteria 4' and 7', which are respectively defined as *Decreasing median residual lifetime* and *Net decreasing median residual lifetime*.

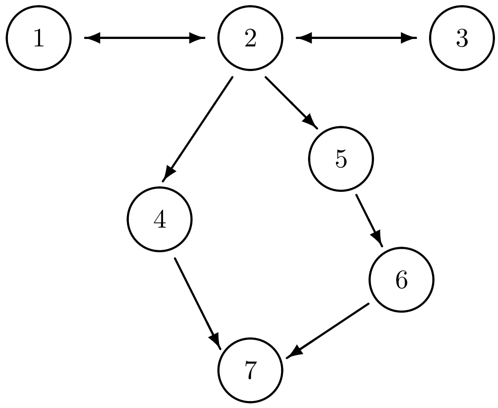

Figure 1: Pattern among criteria for system aging

4 Mean and median residual life functions

Another aspect of interest related to the median residual life function is its relation with the mean residual life function. We are interested in studying if Criterion 4 implies Criterion 4' or viceversa. Also, we want to explore if increasing mean residual life function implies increasing median residual life function or viceversa, but it will be showed that any of the above implications are true.

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